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SOME APPLICATIONS OF THE CHAIN METHOD IN SOLVING A LINEAR DIFFERENCE EQUATION OF FINITE ORDER

Kruglov Viktor
c. ph.-m. s., prof., professor of the Department of Methods of Mathematical Physics
Odessa I.I. Mechnikov National University,
Odessa, str. Dvoryanska, 2, Ukraine, 65029

Abstract. *A scheme of the chain method for solving a finite linear difference equation given in this paper, and a formula for this equation`s general solution of is given. As a result, the formula for the general solution of a difference equation with constant coefficients is given. This formula depend entirely only on the coefficients of this equation. Considered solutions of linear differential equations in the form of a generalized power series, the coefficients of which are found by the chain method. As a result of permuting the elements of the power series, the solution of the equation contains a new function, namely: a hypergeometric function of the fractional order.*

Keywords: *chain, difference equation, hypergeometric function of fractional order.*

Introduction. Differential equations are a powerful tool for modeling and solving problems in many branches of science. Of particular importance in the study of differential equations are difference equations, which describe the behavior of discrete systems. Physical, economic, and social processes evolve over time and are dynamic processes characterized by their speed. Therefore, many of them are described by differential equations. Difference equations arise, for example, in discrete-time models of economic dynamics. Difference equations have been studied in detail in [1 – 6]. Possible applications of such differential equations are described in [7]. These equations can be used to calculate certain definite integrals.

Main text.

1. Scheme of the chain method.

Consider a linear difference equation of order $\boldsymbol{n}$:

$$\boldsymbol{l_{n+k} = a_{1k} l_{n+k-1} + a_{2k} l_{n+k-2} + \cdots + a_{nk} l_{k},\ \ a_{nk} \neq 0,\ \ k = 0, 1, \ldots,} \quad (1)$$

where $\boldsymbol{a_{1k}}$, $\boldsymbol{a_{2k}}$, …, $\boldsymbol{a_{nk}}$ are known functions of integer argument $\boldsymbol{k}$.

A step-by-step solution of equation (1) is used, i.e. at each subsequent step of using equation (1), the solutions found in the previous steps are taken into account, and writing the coefficients of equation (1) in the form a_{jk} does not allow establishing the regularities that exist between these coefficients. Therefore, we denote

$$a_{jk} = a_{n+k-j}^{(j)},\ \ j = \overline{1, n}\,.$$

Then equation (1) takes the form

$$l_{n+k} = a_{n+k-1}^{(1)} l_{n+k-1} + a_{n+k-2}^{(2)} l_{n+k-2} + \cdots + a_k^{(n)} l_k, \qquad k = 0, 1, \dots\ . \tag{2}$$

The general solution of equation (2) is given by the formula

$$l_{n+k} = \varphi_{n-1,n+k} l_{n-1} + \varphi_{n-2,n+k} l_{n-2} + \cdots + \varphi_{0,n+k} l_0,\ \ k = 0, 1, \dots\ , \tag{3}$$

where l_{n-1}, l_{n-2}, …, l_0 are arbitrary constants, integer functions $\varphi_{i,n+k}$, $i = \overline{1, n-1}$, form the fundamental system of solutions of equation (2), and which are constructed in explicit form. Their structure is determined by the formulas

$$\begin{aligned}
&\varphi_{0,n+k} = a_0^{(n)} f_{k,n+k}\,, \\
&\varphi_{1,n+k} = a_1^{(n-1)} f_{k,n+k} + a_1^{(n)} f_{k-1,n+k}\,, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\varphi_{i,n+k} = a_i^{(n-i)} f_{k,n+k} + a_i^{(n-i+1)} f_{k-1,n+k} + \cdots + a_i^{(n)} f_{k-i,n+k}\,, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\varphi_{n-1,n+k} = a_{n-1}^{(1)} f_{k,n+k} + a_{n-1}^{(2)} f_{k-1,n+k} + \cdots + a_{n-1}^{(n)} f_{k-n+1,n+k}\,.
\end{aligned} \tag{4}$$

The functions $f_{k,n+k}$, $f_{k-1,n+k}$, …, $f_{k-p,n+k}$, $p = \min(n, k)$, are related by the equality

$$f_{k,n+k} = a_n^{(1)} f_{k-1,n+k} + a_n^{(2)} f_{k-2,n+k} + \cdots + a_n^{(p)} f_{k-p,n+k}\,, \tag{5}$$

with $f_{0,n+k} \equiv 1$, $f_{j,n+k} \equiv 0$, if $j < 0$.

In constructing the function $f_{k,n+k}$, those coefficients of equation (2) that appear at the k-th step of the recursion are used, namely

$$M_{n,n+k} =$$

$$= \left(a_n^{(1)}, a_{n+1}^{(1)}, \dots, a_{n+k-1}^{(1)};\ a_n^{(2)}, a_{n+1}^{(2)}, \dots, a_{n+k-2}^{(2)}; \dots;\ a_n^{(s)}, a_{n+1}^{(s)}, \dots, a_{n+k-s}^{(s)}\right),$$

where $s = \min(k, n)$.

The elements $a_i^{(j)}$, $j = \overline{1, n}$, are called elements of rank j. The elements of the set $M_{n,n+k}$ that have the smallest subscripts, i.e. $a_n^{(1)}$, $a_n^{(2)}$, …, $a_n^{(s)}$, are called the initial elements of the set, and the elements whose sum of the lower and upper indices is equal to $\boldsymbol{n+k}$ are called the final elements of the set [6]. The element $\boldsymbol{a_{n+k-s}^{(s)}}$ can be both initial and final.

The results of recursion produce products formed by elements of the set $\boldsymbol{M_{n,n+k}}$, which have the following structure: two arbitrary adjacent factors in each of the products satisfy the following multiplication rule:

$$\boldsymbol{\cdots\ a_{k}^{(i_1)} a_{k+i_1}^{(i_2)} \cdots, \qquad i_1, i_2 = \overline{1, n}\,,}$$

where $\boldsymbol{i_1}$ and $\boldsymbol{i_2}$ can be equal to each other.

A chain consisting of elements of the set $\boldsymbol{M_{n,n+k}}$ is called the product of the maximum possible number of elements from this set, but for two arbitrary adjacent factors in this product the specified multiplication rule must be used. It follows that an arbitrary chain starts with some initial element of the set $\boldsymbol{M_{n,n+k}}$ and ends with one of the final elements of this set.

The structure of an arbitrary chain has the following form:

$$\boldsymbol{a_{n}^{(i_1)} a_{n+i_1}^{(i_2)} a_{n+i_1+i_2}^{(i_3)} \cdot \ldots \cdot a_{n+i_1+i_2+\cdots+i_{r-1}}^{(i_r)}\,,}$$

$$\boldsymbol{n + i_1 + \cdots + i_{r-1} + i_r = n + k\,.}$$

The order of a chain is the sum of the ranks of all the factors that form this chain. From the elements of the set $\boldsymbol{M_{n,n+k}}$ it is possible to form chains of order $\boldsymbol{k}$ only, because

$$\boldsymbol{i_1 + \cdots + i_{r-1} + i_r = k\,.}$$

The function $\boldsymbol{f_{k,n+k}}$ is the sum of all chains of order $\boldsymbol{k}$ that can be composed of elements of the set $\boldsymbol{M_{n,n+k}}$. Let us now count the number of terms in the function $\boldsymbol{f_{k,n+k}}$. Let a chain of order $\boldsymbol{k}$ have $\boldsymbol{x_1}$ elements of rank one, $\boldsymbol{x_2}$ elements of rank two, etc., $\boldsymbol{x_n}$ elements of rank $\boldsymbol{n}$. Then

$$\boldsymbol{k = x_1 + 2x_2 + \cdots + nx_n\,.}$$

It follows that the number of all chains of order $\boldsymbol{k}$ that can be composed of elements of the set $\boldsymbol{M_{n,n+k}}$ is equal to

$$\boldsymbol{Q_k^{(n)} = \sum_{x_1+2x_2+\cdots+nx_n=k} \frac{(x_1 + x_2 + \cdots + x_n)!}{x_1!\, x_2! \ldots x_n!}\,.}$$

The function $\boldsymbol{f_{k-1,n+k}}$ is constructed similarly. It is the sum of chains of order $\boldsymbol{k-1}$, which are formed from the elements of the set

$$\boldsymbol{M_{n+1,n+k}} = \left(\boldsymbol{a_{n+1}^{(1)}, \dots, a_{n+k-1}^{(1)};\ a_{n+1}^{(2)}, \dots, a_{n+k-2}^{(2)}; \dots;\ a_{n+1}^{(p)}, \dots, a_{q}^{(p)}}\right),$$

and the number of such chains is

$$\boldsymbol{Q_{k-1}^{(n)}} = \sum_{x_1+2x_2+\dots+nx_n=k-1} \frac{(\boldsymbol{x_1+x_2+\dots+x_n})!}{\boldsymbol{x_1!\,x_2!\dots x_n!}},$$

etc.. The number of chains of order $\boldsymbol{k-m}$ (i.e., terms in the function $\boldsymbol{f_{k-m,n+k}}$) formed from the elements of the set

$$\boldsymbol{M_{n+m,n+k}} = \left(\boldsymbol{a_{n+m}^{(1)}, \dots, a_{n+k-1}^{(1)};\ a_{n+m}^{(2)}, \dots, a_{n+k-2}^{(2)}; \dots;\ a_{n+m}^{(p)}, \dots, a_{q}^{(p)}}\right),$$

is equal to

$$\boldsymbol{Q_{k-m}^{(n)}} = \sum_{x_1+2x_2+\dots+nx_n=k-m} \frac{(\boldsymbol{x_1+x_2+\dots+x_n})!}{\boldsymbol{x_1!\,x_2!\dots x_n!}}.$$

The solution to equation (2) is given by formulas (3) and (4).

Let

$$\boldsymbol{a_{jk} = a_{n+k-j}^{(j)} \equiv a_j,\ \ k = 0, 1, \dots,}$$

that is, we are dealing with a difference equation with constant coefficients

$$\boldsymbol{l_{n+k} = a_1 l_{n+k-1} + a_2 l_{n+k-2} + \dots + a_n l_k,\ \ k = 0, 1, \dots.} \tag{6}$$

Now, in any chain, the order of multiplication of its elements according to the specified rule loses its meaning, and the chain structure takes the form

$$\boldsymbol{a_1^{x_1} a_2^{x_2} \dots a_n^{x_n}}.$$

Then

$$\boldsymbol{f_{k,n+k}} = \sum_{x_1+2x_2+\dots+nx_n=k} \frac{(\boldsymbol{x_1+x_2+\dots+x_n})!}{\boldsymbol{x_1!\,x_2!\dots x_n!}} \cdot \boldsymbol{a_1^{x_1} a_2^{x_2} \dots a_n^{x_n} = R_k^{(n)}},$$

$$\boldsymbol{f_{k-m,n+k}} = \sum_{x_1+2x_2+\dots+nx_n=k-m} \frac{(\boldsymbol{x_1+x_2+\dots+x_n})!}{\boldsymbol{x_1!\,x_2!\dots x_n!}} \cdot \boldsymbol{a_1^{x_1} a_2^{x_2} \dots a_n^{x_n} = R_{k-m}^{(n)}}.$$

Equation (5) can be written in the following way:

$$\begin{aligned} \boldsymbol{R_k^{(n)}} &= \boldsymbol{a_1 R_{k-1}^{(n)} + a_2 R_{k-2}^{(n)} + \dots + a_n R_{k-n}^{(n)}}, \quad \boldsymbol{k > n}; \\ \boldsymbol{R_k^{(n)}} &= \boldsymbol{a_1 R_{k-1}^{(n)} + \dots + a_{k-1} R_1^{(n)}}, \quad \boldsymbol{k = \overline{1,n}}; \end{aligned} \tag{7}$$

$$R_0^{(n)} = 1, R_1^{(n)} = a_1 . \qquad (8)$$

Equation (4) can be written in the following way:

$$\varphi_{0,n+k} = a_n R_k^{(n)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\varphi_{i,n+k} = a_{n-i} R_k^{(n)} + \cdots + a_n R_{k-i}^{(n)}, \quad i = \overline{1, n-2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\varphi_{n-1,n+k} = R_{k+1}^{(n)} = a_1 R_k^{(n)} + \cdots + a_n R_{k-n+1}^{(n)} .$$

The general solution of equation (6) is given by the formula

$$l_{n+k} = \sum_{i=1}^{n} (a_i R_k^{(n)} + a_{i+1} R_{k-1}^{(n)} + \cdots + a_n R_{k-n+i}^{(n)}) l_{n-i} , \quad k = 0, 1, \ldots , \qquad (9)$$

where the numbers $R_k^{(n)}$, if $k = \overline{1, n}$ or $k > n$, are calculated by formulas (7), (8).

It can be proven that among the solutions (9) there are known solutions of the form $k^m \lambda^s$, where λ is the root of the characteristic equation

$$\lambda^n = a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n .$$

In [1] an example is given where the equation is considered

$$l_{k+2} = a l_{k+1} + b l_k .$$

Its solution is given by the formula

$$l_{k+2} = (a l_1 + b l_0) \sum_{m=0}^{[k/2]} C_{k-m}^{m} a^{k-2m} b^m + b l_1 \sum_{m=0}^{[k/2]} C_{k-m-1}^{m} a^{k-2m-1} b^m .$$

This formula implies:

1) if $\lambda_1 \neq \lambda_2$ and $l_1 = \lambda_1$, $l_0 = 1$, then

$$l_{k+2} = \lambda_1^{k+2};$$

2) if

$$\lambda_1 = \lambda_2 = \frac{a}{2}, \qquad l_1 = \lambda_1 = \frac{a}{2}, \qquad l_0 = 1,$$

then

$$l_{k+2} = \left(\frac{a}{2}\right)^{k+2} \lambda_1^{k+2},$$

and for $l_1 = \lambda_1$, $l_0 = 0$

$$l_{k+2} = (k+2) \lambda_1^{k+2}.$$

2. Application of the chain method to solving linear differential equations.

Differential equations in the complex plane are studied, respectively, in [3], [4], [5]

$$t^2(A_1t^2+B_1t+C_1)u''+t(A_2t^2+B_2t+C_2)u'+(A_3t^2+B_3t+C_3)u=0, \quad C_1\neq 0, \tag{10}$$

$$\sum_{i=1}^{n}(A_it^2+B_it+C_i)\,t^iu^{(i)}=0, \qquad C_n\neq 0, \tag{11}$$

where A_i, B_i and C_i are known numbers (real or complex);

$$t^2P_1(t)u''+tP_2(t)u'+P_3(t)u=0, \tag{12}$$

where the functions $P_i(t)$, $i=\overline{1,3}$, are analytic in a neighborhood of the Fuchsian zero point. For these functions, the expansions in the specified neighborhood take place

$$P_i(t)=a_{i_0}+a_{i_1}t+a_{i_2}t^2+\cdots\ , \qquad a_{i_0}\neq 0. \tag{13}$$

When solving these equations, a single approach is used, namely: the solution of the equation in the vicinity of the Fuchs point $t=0$ is sought in the form of a generalized power series (Frobenius solution)

$$u(t)=t^{\rho}(l_0+l_1t+l_2t^2+\cdots), \qquad l_0\neq 0, \tag{14}$$

where the parameters ρ, l_i need to be found.

It is known [8] that this series absolutely converges in the ring $0<|t|<R$, where R is the distance from the point $t=0$ to the nearest singular point of the differential equation.

Conclusions.

For equations (10) – (12) with respect to the parameters l_i, third-order difference equations were obtained. The chain method is used to solve these difference equations. Further, taking into account the rather cumbersome structure of the parameters l_i, and in order to make the series (14) more transparent, we rearrange the members of the series. As a result, it became possible to write the series (14) as a linear combination of standard hypergeometric series and the hypergeometric series of fractional order introduced in these works, namely

$$F_{q/k}(a_1,a_2;1,b_1;t)=\sum_{m=0}\frac{(a_1+q/k)_m(a_2+q/k)_m}{(1+q/k)_m(b_1+q/k)_m}t^m\ , \quad q=\overline{1,k-1},$$

where $$(a)_m=a(a+1)+\cdots+(a+m-1).$$

The next task is to study the properties of the hypergeometric series of fractional order.

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Анотація. *В цій роботі наведено схему методу ланцюгів стосовно розв'язання скінченного лінійного різницевого рівняння, і приведено формулу загального розв'язку цього рівняння. Як наслідок, наведено формулу загального розв'язку різницевого рівняння зі сталими коефіцієнтами, яка цілком залежить тільки від коефіцієнтів цього рівняння. Розглянуті розв'язки лінійних диференціальних рівнянь у вигляді узагальненого степеневого ряду, коефіцієнти якого знаходяться методом ланцюгів. Внаслідок перестановки елементів степеневого ряду розв'язок рівняння містить нову функцію, а саме: гіпергеометричну функцію дробового порядку.*

Ключові слова: *ланцюг, різницеве рівняння, гіпергеометрична функція дробового порядку.*

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