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LAMB WAVE MODE SEPARATION IN LAMINATED BEAMS

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Abstract. *This study presents analytical partitioning theories for one-dimensional delamination in laminated composite beams, developed using Euler and Timoshenko beam models. These theories are compared with numerical simulations based on Timoshenko beam formulations and four-node quadrilateral elements. One major goal is to derive orthogonal pure modes for laminated beams and calculate energy release rate partitions accordingly, including the subtle yet important consideration of stealthy interaction – an energy exchange between fracture modes within the beam structure that does not alter the total energy release rate.*

The proposed framework also examines the applicability of partitioning models under general loading conditions, such as axial forces, bending moments, and shear forces. Explicit relationships for averaged partition rules are formulated in the context of two-dimensional elasticity, informed directly by beam theory. This work aims to offer both clarity and accuracy in delamination analysis, contributing to improved failure prediction and structural design in layered composite systems.

Key words: *delamination, mode separation, Lamb waves, laminated composite beams, mixed-mode partition.*

Introduction.

In recent decades, the widespread use of layered materials in advanced engineering applications has significantly reshaped the landscape of structural design and analysis. From aerospace components to biomedical devices, these materials–often consisting of distinct interfaces between layers–offer remarkable strength-to-weight ratios and tailored mechanical properties. However, their structural performance is intricately tied to the behavior of these interfaces. Fiber-reinforced composite laminates, a prominent class of layered materials, exemplify this complexity. Due to their heterogeneous and anisotropic nature, they present considerable challenges for mechanical analysis and long-term durability assessment.

One of the most critical and prevalent failure mechanisms in composite laminates is delamination – a global-scale fracture that propagates along the interfaces between plies [1]. This interfacial failure not only undermines the load-bearing capacity of the structure but also drastically compromises its integrity. Understanding the conditions under which delamination initiates and propagates is therefore vital for the safe design and maintenance of structures composed of such materials. Central to this understanding is the concept of fracture partitioning, mode which describes how the total energy release rate at a crack tip is distributed among different modes of deformation: opening, shearing, and tearing [2].

Recent research has focused intensively on developing fracture theories that can reliably predict delamination behavior [3]. The authors of this study have previously contributed a comprehensive body of work on one-dimensional fracture modeling in beams and circular plates composed of isotropic materials. These studies have laid the groundwork for analyzing practical delamination scenarios, many of which can be approximated as one-dimensional problems where cracks propagate primarily in a single direction.

Examples of one-dimensional delamination phenomena include through-width delaminations in straight or curved composite beams, circular ring-type delaminations in laminated plates and shells caused by drilling, and material separation in biological tissues under puncture. Other cases involve the separation of stiffeners from skins in stiffened panel structures, or the debonding of layers in multi-material assemblies subjected to complex loading. Among all the configurations studied, the double cantilever beam (DCB) stands out as the most fundamental and widely analyzed setup for investigating one-dimensional fracture [4].

In recent publications, researchers have rigorously derived and validated one-dimensional fracture theories using numerical simulations. These models have demonstrated strong predictive capability and have provided insight into the energy release rate and partition behavior under various loading conditions. Additionally, experimental data has been employed to benchmark the performance of the developed theories and compare them against other partitioning approaches, solidifying the credibility of the framework.

Although foundational, later studies identified limitations in limitation rules and proposed improved formulations using combined numerical and analytical approaches based on stress intensity factors. These refinements helped correct inaccuracies and offered more reliable predictions under mixed-mode loading conditions. Analytical models were developed for bi-layered laminated composite plates, which correctly identified several orthogonal pure modes. However, a critical phenomenon known as stealthy interaction was overlooked in these approaches. This subtle transfer of energy between modes–while conserving total energy–plays a vital role in mixed-mode fracture analysis. Neglecting this interaction has led to errors in fracture mode partitioning and demonstrates the importance of capturing coupling effects in analytical theories. Some studies expanded the scope of partition theories to general non-homogeneous laminated composites. In doing so, delamination was modeled as a crack within an equivalent anisotropic body, effectively smoothing out the material interface to recover the characteristic square-root singular stress field at the crack tip.

In this paper, theoretical mode partitioning models for one-dimensional delamination in laminated composite beams based on the Euler and Timoshenko beam theories are presented. These partitioning theories are compared with the results of numerical simulations performed using the Timoshenko beam model and four-node square elements. Among the objectives of the paper are to derive orthogonal pure modes for laminated composite beams; calculate energy release rate partitions using these orthogonal pure modes; and take into account the latent interaction when using Euler beams made of laminated composite. It is intended to analyze the applicability of the theories to general loading conditions including axial forces, bending moments, and shear forces, and to derive explicit relations for "average partitioning rules" for energy release rates in the context of two-dimensional elasticity, directly based on the beam partitioning theory.

Lamb wave separation method.

Lamb wave separation is considered on a model of a laminated loaded double cantilever beam with a delamination of length a. It is convenient to use two different longitudinal axes x and n. The origin of both axes is at the crack tip at fixed point. A positive bending moment creates a positive curvature. The contact between the upper and lower sublayers is not considered. The Euler theory relationships for composite laminated beams fixes for the strain energy per unit length, U0, the following equation

$$U = \frac{1}{2b} \begin{Bmatrix} N(x) \\ -M(x) \end{Bmatrix}^T \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N(x) \\ -M(x) \end{Bmatrix}, \quad (1)$$

where

- $N(x)$, $M(x)$ are the axial force and bending moments respectively;
- b is the width of the beam;
- A, B, D are the equivalent extensional, coupling and bending stiffness.

The total energy release rate G is defined as

$$G = \frac{\partial U}{\partial A}. \quad (2)$$

There are two sets of orthogonal pure modes in the presence of deformation in the form of a crack. The first set corresponds to zero relative shear displacement immediately behind the crack tip (mode I) and zero crack opening force in front of its tip (mode II). The second set corresponds to zero relative opening immediately behind the crack tip (mode II) and zero shear force at the crack tip (mode I). The calculation model assumes a preliminary description of modes with zero relative displacement. The second stage of calculations includes a description of modes with zero force, taking into account the orthogonality condition. At the third stage, modes I from the first set are derived, corresponding to zero relative shear displacement. In this case, the Euler theory of laminated composite beams is used

$$\begin{Bmatrix} N_{1,2}(x)/b \\ -M_{1,2}(x)/b \end{Bmatrix} = \begin{Bmatrix} N_{1,2B}/b \\ (P_{1,2B}x - M_{1,2B})/b \end{Bmatrix} = \begin{bmatrix} A_{1,2} & B_{1,2} \\ B_{1,2} & D_{1,2} \end{bmatrix}. \quad (3)$$

Since there are four independent variables, four modes are required for the mixed-mode partition. Any combination of pure modes can be chosen and the same mixed-mode partition will be obtained. However, since modes m_{b1}, m_{b2}, and m_{b3}, by definition, do not contribute to the crack opening force F_{nB}, it is convenient to choose the pure Lamb wave modes m_{a1}, m_{a2}, m_{a3}, and m_{a4}. Therefore, F_{nB} is simply defined as the crack opening force from pure mode

$$\begin{Bmatrix} M_{1B} \\ M_{2B} \\ N_{1B} \\ N_{2B}0 \end{Bmatrix} = [m_{a1}, m_{a2}, m_{a3}, m_{a4}] \cdot \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}, \tag{4}$$

where a_1, a_2, a_3, a_4 are the mode partition coefficients.

According to the Euler beam theory, the basic assumption is that the out-of-plane shear moduli are infinitely large. A consequence of this assumption is the existence of a latent interaction between each orthogonal pair of pure modes. It should be noted that the concept of latent interaction involves the flow of energy from one mode to another while maintaining the total amount of energy. To partition the mixed modes, it is necessary to define this interaction. The mode I component of the energy release rate is expressed using the relation

$$G_I = \frac{F_{nB} D_p}{2b\delta a}, \tag{5}$$

where D_p is the relative opening at an infinitely small distance. This force is defined as

$$F_{nB} = \int_0^{\delta a} \sigma_n dl, \tag{6}$$

where variable l fixed the origin at the crack tip.

Provided that the out-of-plane shear moduli are finite, the shear forces cause shear deformation. However, this relationship requires a separate study. Two shear forces act at the crack tip of two sub-laminates. It is necessary to determine the numerical value of the shear stiffness H of the laminated structure. It is also necessary to take into account the assumptions about the conditions of plane stress, plane strain or mixed conditions. The dependencies for N, M, P can be written in this case as follows

$$\begin{Bmatrix} N_{1,2}(x)/b \\ -M_{1,2}(x)/b \\ P_{1,2}(x)/b \end{Bmatrix} = \begin{bmatrix} A_{1,2} & B_{1,2} & 0 \\ B_{1,2} & D_{1,2} & 0 \\ 0 & 0 & H_{1,2} \end{bmatrix} \begin{Bmatrix} du_{1,2}/dx \\ -d\psi_{1,2}/dx \\ du_{1,2}/dx - \psi_{1,2} \end{Bmatrix}, \tag{7}$$

where

- u is the axial displacement;

- ψ is the cross-sectional rotation function.

Mixed mode partitioning theories can be constructed based on the Euler and Timoshenko beam theories. In the context of the Timoshenko beam theory, there is no interaction between the m_{ai} modes and the m_{bi} modes (i = 1, 2, 3). On the other hand, in the Euler beam theory, there is complete interaction between these modes. Most studies assume that the Euler and Timoshenko partitioning theories represent upper and lower bounds on the possible partitions. Numerical simulations on a wide range of different isotropic and laminated composite beams have confirmed this hypothesis. Therefore, it is reasonable to expect that the mean between the two theories, i.e., the half-interaction, will yield results comparable to the partition obtained using the two-dimensional finite element method (2D FEM). This mean partitioning rule yields the following partitioning options

$$G_I = a_1^2 G_1 + a_1 a_2 \Delta G_{12} / 2 + a_1 a_3 \Delta G_{13} / 2 + a_2 a_3 \Delta G_{23} / 2, \quad (8)$$

$$G_{II} = b_1^2 G_1 + b_1 b_2 \Delta G_{12} / 2 + b_1 b_3 \Delta G_{13} / 2 + b_2 b_3 \Delta G_{23} / 2, \quad (9)$$

where a, b are the pure mode I and II coefficients.

We consider subsurface delamination ($h_2/h_1 \neq 1$). This case is typical for homogeneous semi-infinite plates, where stress can be described using stress intensity factors. This method can be extended to heterogeneous systems, which is important for studying such phenomena as delamination of thin films from substrates and delamination in composite materials. The effect of shear across the thickness is ignored, since the spalling region is thin. The partitions obtained using the Euler and Timoshenko beam theories, as well as the rules of averaged partitioning, have the following form

$$G_I = \frac{3M_{1B}\left(2M_{1B} - h_1 N_{1B}\right)}{b^2 h_1^3 E_1}, \quad (10)$$

$$G_{II} = \frac{N_{1B}\left(6M_{1B} + h_1 N_{1B}\right)}{2b^2 h_1^2 E_1}, \quad (11)$$

where E_1 is the Yoing's modulus of the spull.

The total energy release rate is

$$G = \frac{12M_{1B}^2 + h_1^2 N_{1B}^2}{2b^2 h_1^3 E_1}. \tag{12}$$

Numerical results for the Timoshenko and Euler theories are shown in Fig. 1.

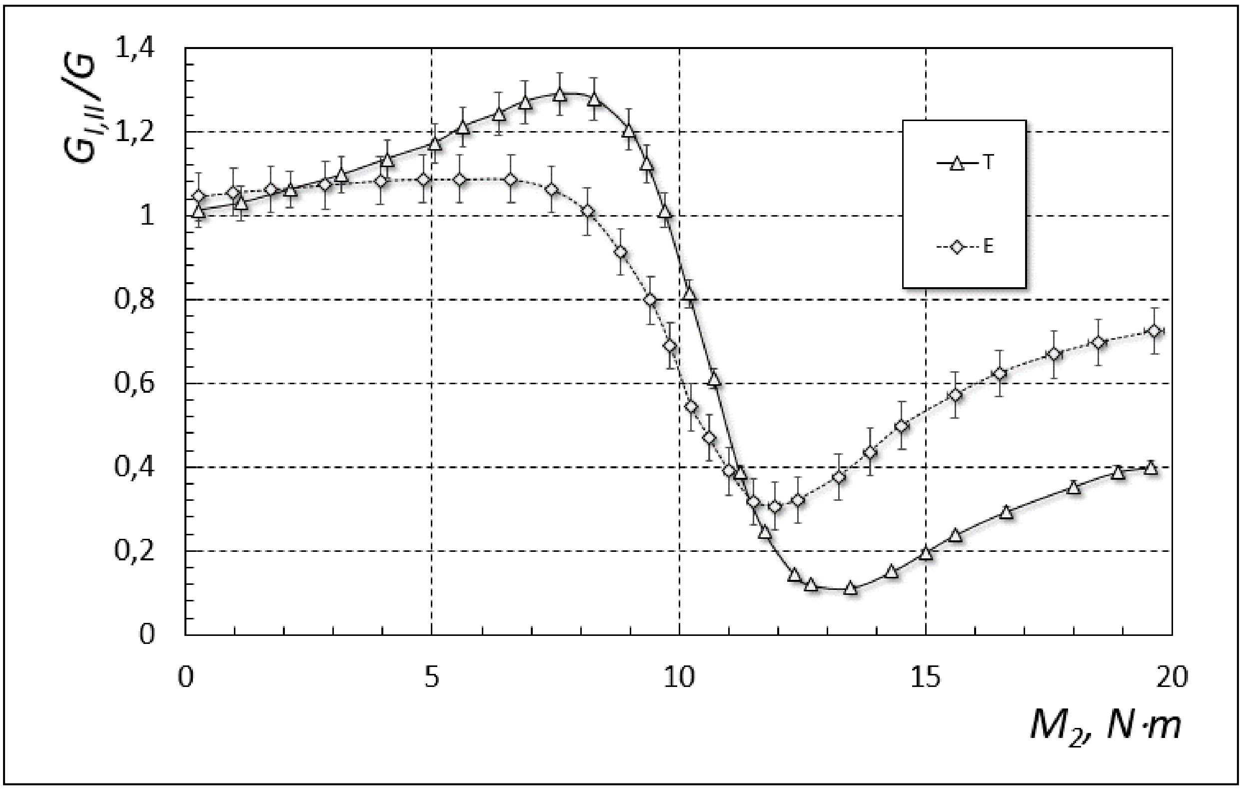

Figure 1 – Comparison between Timoshenko (T) and Euler (E) theories

Summary and conclusions.

In this paper, fully analytical theories of partitioning by modes based on the Euler and Timoshenko beam theories for laminated composites were developed. For the Euler beam theory and the Timoshenko beam theory, the first group of pure modes ma, mb is the same. In the Euler beam theory, there is a second group of pure modes, which differs from the first. And in the Timoshenko beam theory, the second group of pure modes coincides with the first, so it is believed that the second set does not exist in this theory. Therefore, it is the first set of pure modes that forms the complete basis for partitioning the mixed mode. There are non-obvious hidden interactions between modes I and II in the Euler beam partition theory. These interactions disappear in the Timoshenko beam partition theory, resulting in the absence of a second set of pure modes in the Timoshenko theory. The mode partitions obtained from the Euler and Timoshenko beam theories agree well with the corresponding results from finite element models (FEMs) of beams. Approximate rules for average partitioning have also been proposed, which are based on defining the minimum points for GI/G and GII/G as pure modes.

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